CORE Metadata, citation and similar papers at core.ac.uk
Provided by Elsevier - Publisher Connector

Generalized Circulants and Class Functions of Finite Groups. II*

David Chillag
Department of Mathematics
Technion — Israel Institute of Technology
Haifa 32000, Israel

Submitted by Thomas J. Laffey

ABSTRACT

In a previous article it was shown that the semisimple algebra of class functions of a finite group gives rise to a family of matrices that enjoys many of the properties of circulant matrices. Varying the groups yields different families (circulants, block circulants of all levels, and others), whose properties are simple consequences of character theory. In this article we observe that many more such families can be constructed by a slight modification of the method used in the first paper. The new set of families includes all the families obtained in the first paper, the families of skew-circulants, $\{k\}$-circulants, retrocirculants, some g-circulants, and many others. Known properties of these specific families are special cases of the properties of the general families constructed, many of which are themselves corollaries of simple facts about the regular representation of semisimple commutative algebras.

INTRODUCTION

Let A be a finite-dimensional semisimple commutative algebra over the complex number field $\mathbb{C}$ (FDSSC algebra, for short). To each pair of bases (Σ, Δ) of A we assign a family of matrices, $M(A;\Sigma,\Delta)$, by regularly representing A with respect to Σ and Δ (see Section I for the precise definition). We write $M(A;\Sigma,\Sigma) = M(A;\Sigma)$.

Let G be a finite group, and $\mathrm{cf}(G)$ the set of all complex class functions of G. Then $\mathrm{cf}(G)$ is a finite-dimensional semisimple commutative $\mathbb{C}$-algebra with a natural inner product: $[\phi,\eta] = (1/|G|)\Sigma_{x\in G}\phi(x)\overline{\eta(x)}$ for all $\phi,\eta \in$

*This research was partially supported by the Fund for the Promotion of Research at the Technion.

© Elsevier Science Publishing Co., Inc., 1988
52 Vanderbilt Ave., New York, NY 10017 0024-3795/88/$3.50

cf(G). For each pair of bases (Σ, Δ) of cf(G) we set $M(G; \Sigma, \Delta) = M(\mathrm{cf}(G); \Sigma, \Delta)$ and $M(G;\Sigma,\Sigma) = M(G;\Sigma)$. Each matrix of $M(G;\Sigma,\Delta)$ [respectively, $M(G;\Sigma)$] is called a $(G;\Sigma,\Delta)$-circulant [respectively, $(G;\Sigma)$-circulant].

In [1] the algebra $M(G) = M(G;\mathrm{Irr}(G))$ was studied, where Irr(G) is the orthonormal basis of cf(G) consisting of all irreducible characters of G. It was shown that circulants and block circulants with circulant blocks of all levels (see [2, p. 188] for definition) and other families are $M(G)$'s for different choices of G's and that their properties are consequences of those of $M(G)$ for a general G (see the introduction of [1] for a brief description of these properties).

In the present article we observe that replacing Irr(G) by other choices of pairs of bases (Σ, Δ) of cf(G), many more circulant-like families can be constructed. In fact skew-circulants, $\{k\}$-circulants (see [2, p. 83–85]), and other families are all $M(G;\Sigma)$'s for various Σ's, and retrocirculants, g-circulants of order n with $(n, g) = 1$ (see [2, p. 155–156]), and other families are all $M(G;\Sigma,\Delta)$'s for various pairs (Σ,Δ) and a cyclic G. A survey of such examples and others is given in Section II. The notation, notions, and basic properties of $(G;\Sigma,\Delta)$-circulants are presented in Section I, and those on block $(G;\Sigma,\Delta)$-circulants in Section III. (Some of the results of Section III follow from the corresponding ones in [1, Section 2], by changing of bases). Most of these properties are consequences of those of $M(A;\Sigma)$ and $M(A;\Sigma,\Delta)$, where A is an FDSSC algebra with bases Σ and Δ. In fact $M(A;\Sigma)$ [and in particular $M(G;\Sigma)$] enjoys many [all for $M(G;\Sigma)$ if Σ is orthonormal] of the properties of $M(G)$. Sections I and III are straightforward generalizations of [1] and look very much like [1]. The benefit of the generalization is the large number of new families of matrices it provides. In the example section (II) we indicate how properties of the examples (many of them can be found in [2]) follow as special cases from the basic observations of Sections I and III.

I. NOTATION AND BASIC PROPERTIES

Our basic notation and notions of matrix theory are taken from [2], and those of group theory from [3].

Let A be a semisimple commutative algebra of dimension k over $\mathbb{C}$, and let (Σ, Δ) be a pair of bases of A. Let $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ and $\Delta = \{\delta_1, \delta_2, \dots, \delta_k\}$. For each $a \in A$ define a $k \times k$ matrix $M(a;\Sigma,\Delta) = \big(m_{ij}(a;\Sigma,\Delta)\big)$ by $a\sigma_i = \sum_{j=1}^k m_{ij}(a;\Sigma,\Delta)\delta_j$ for $i, j = 1,2,\dots,k$. Note that if Δ is orthonormal with respect to an inner product $[\ ,\ ]: A \to \mathbb{C}$, then $m_{ij}(a;\Sigma,\Delta) = [a\sigma_i, \delta_j]$ for all $i, j = 1,2,\dots,k$. In fact, if we define for all

$a \in A$ the linear transformation $T_a: A \to A$ by $T_a(b) = ab$ for all $b \in A$, then the matrix of T_a with respect to the pair (Σ, Δ) is $[M(a; \Sigma, \Delta)]^t$. Clearly, $\{T_a | a \in A\}$ is a $\mathbb{C}$-algebra isomorphic to A, the map $a \to T_a$ being an algebra isomorphism. We set $M(A; \Sigma, \Delta) = \{M(a; \Sigma, \Delta) | a \in A\}$ and write $M(a; \Sigma) = M(a; \Sigma, \Sigma)$, $M(A; \Sigma) = M(A; \Sigma, \Sigma)$.

The k^3 numbers a_{ijr} which are defined by $\sigma_i \delta_j = \sum_{r=1}^{k} a_{ijr} \delta_r$ will be called the (Σ, Δ)-constants of A.

By [4, p. 13] there is a basis $\mathscr{E} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ of A such that each ϵ_i is a common eigenvector for all T_a, $a \in A$. Denote by $a(1), a(2), \ldots, a(k)$ the eigenvalues of T_a, $a \in A$, ordered so that $a\epsilon_i = a(i)\epsilon_i$ for all $a \in A$ and $i = 1, 2, \ldots, k$. In particular $\epsilon_i \epsilon_j = \epsilon_i(j)\epsilon_j = \epsilon_j(i)\epsilon_i$, which implies that $\epsilon_i \epsilon_j = 0$ and $\epsilon_i(j) = 0$ for $i \neq j$. As $\epsilon_i^2 = \epsilon_i(i)\epsilon_i \neq 0$ (A is semisimple) we may replace ϵ_i with another one such that $\epsilon_i^2 = \epsilon_i$. Thus , A has a basis $\mathscr{E} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ satisfying the following three conditions: (1) $a\epsilon_i = a(i)\epsilon_i$ for all $a \in A$ and $i = 1, 2, \ldots, k$; (2) $\epsilon_i \epsilon_j = 0$ for $i \neq j$; (3) $\epsilon_i^2 = \epsilon_i$ for $i = 1, 2, \ldots, k$. Any such basis will be called a standard basis. Fix a standard basis $\mathscr{E} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ of A, and let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be any other basis. Define the matrix $X(A; \Sigma) = (x_{ij}(A; \Sigma))$ by $\sigma_i = \sum_{i=1}^{k} x_{ij}(A; \Sigma)\epsilon_j$ for all $i = 1, 2, \ldots, k$. So $[X(A; \Sigma)]^t$ is the transition matrix from $\mathscr{E}$ to Σ.

REMARKS.

(i) It is well known that a semisimple commutative algebra of finite dimension r, over an algebraically closed field K is isomorphic to a direct sum of r copies of K. Thus every FDSSC algebra is determined up to isomorphism by its dimension. Our emphasis here lies in the different appearances of the representing matrices as we vary the bases. This will yield different circulant-like families of matrices. See the remark on the noncommutative case at the end of this section.

(ii) The ϵ_i's mentioned above are, in fact, just the primitive orthogonal idempotents of A (they are automatically central here).

PROPOSITION 1.1. *Let A be a commutative semisimple $\mathbb{C}$-algebra of dimension k, $\mathscr{E}$ a fixed standard basis, and Σ and Δ two other bases. Let P^t be the transition matrix from Δ to Σ. Then*:

(a) *Set* $X = X(A; \Sigma)$. *Then* $X^{-1}M(a; \Sigma)X = \operatorname{diag}(a(1), a(2), \ldots, a(k))$ *for all* $a \in A$.

(b) $M(a; \Sigma, \Delta) = M(a; \Sigma)P$ *for all* $a \in A$.

(c) $M(a; \Sigma) = PM(a; \Delta)P^{-1}$ *for all* $a \in A$.

(d) $X(A; \Sigma) = PX(A; \Delta)$.

(e) *The map* $a \to M(a; \Sigma)$ *is an algebra isomorphism from* A *to* $M(A; \Sigma)$. *The set* $\{M(\sigma; \Sigma) | \sigma \in \Sigma\}$ *is a basis of* $M(A; \Sigma)$. *Furthermore, if* $M =$

$\sum_{\sigma \in \Sigma} a_\sigma M(\sigma; \Sigma)$ *is an arbitrary matrix of* $M(A; \Sigma)$, *then the eigenvalues of* M *are* $\sum_{\sigma \in \Sigma} a_\sigma \sigma(i)$, $i = 1, 2, \dots, k$.

Proof. Let $a \in A$. As $\mathscr{E}$ is standard, we have that $M(a; \mathscr{E}) = \operatorname{diag}(a(1), a(2), \dots, a(k))$. Now, $[M(a; \mathscr{E})]^t$, $[M(a; \Sigma)]^t$, $[M(a; \Delta)]^t$, and $[M(a; \Sigma, \Delta)]^t$ are representing matrices of T_a relative to the bases $\mathscr{E}, \Sigma, \Delta$, (Σ, Δ) respectively. Thus, $[M(a; \Sigma)]^t = (X^t)^{-1}[M(a; \mathscr{E})]^t X^t = (P^t)^{-1}[M(a; \Delta)]^t P^t$, $[M(a; \Sigma, \Delta)]^t = P^t[M(a; \Sigma)]^t$, and $[X(A; \Sigma)]^t = [X(A; \Delta)]^t P^t$, from which (a), (b), (c), and (d) follow. The map $a \to M(a; \Sigma)$ is the composition of three algebra isomorphisms:

$$A \xrightarrow{f_1} \{T_a | a \in A\} \xrightarrow{f_2} \left\{M^t | M \in M(A; \Sigma) \xrightarrow{f_3} M(A; \Sigma)\right\},$$

where f_1 is the regular representation of A, $f_2(T_a) = [M(a; \Sigma)]^t$ is the matrix representation of T_a relative to Σ, and $f_3(M^t) = M$. Note that as $\{M^t | M \in M(A; \Sigma)\}$ is commutative, f_3 is multiplicative. The rest of the lemma follows from this and part (a). ■

REMARK. The map $a \to M(a; \Sigma, \Delta)$ is a vector-space isomorphism from A onto $M(A; \Sigma, \Delta)$.

Let G be a finite group. Denote the conjugacy classes of G by $C_1, C_2, \dots, C_k$. For each $i = 1, 2, \dots, k$, let $\epsilon_i \in \operatorname{cf}(G)$ be defined by $\epsilon_i(C_j) = \delta_{ij}$. Then $\mathscr{E} = \{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}$ is a standard basis for $\operatorname{cf}(G)$. Clearly $\phi\epsilon_i = \phi(C_i)\epsilon_i$ for all $\phi \in \operatorname{cf}(G)$ [here $\phi(C_i) = \phi(i)$ is the value of ϕ on C_i]. It follows that if $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ is any basis of $\operatorname{cf}(G)$, then $X(G; \Sigma) = X(\operatorname{cf}(G); \Sigma) = (\sigma_i(C_j))$. This matrix, $X(G; \Sigma)$, is called the Σ-table of G. The character table of G is in fact $X(G; \operatorname{Irr}(G))$. Recall that elements of $M(G; \Sigma) = M(\operatorname{cf}(G); \Sigma)$ are called (G, Σ)-circulants and those of $M(G; \Sigma, \Delta) = M(\operatorname{cf}(G); \Sigma, \Delta)$ are called $(G; \Sigma, \Delta)$-circulants.

COROLLARY 1.2. *Let G be a finite group, $C_1, C_2, \dots, C_k$ its conjugacy classes, Σ and Δ two bases of* $\operatorname{cf}(G)$, *and P^t the transition matrix from Δ to Σ. Then $M(G; \Sigma)$ is a commutative $\mathbb{C}$-algebra isomorphic to* $\operatorname{cf}(G)$ *via the algebra isomorphism $\phi \to M(\phi; \Sigma)$. Moreover:*

(a) *Set $X = X(G; \Sigma)$. Then $X = PX(G; \Delta)$ and $X^{-1}M(\phi; \Sigma)X = \operatorname{diag}(\phi(C_1), \phi(C_2), \dots, \phi(C_k))$ for all $\phi \in$* $\operatorname{cf}(G)$.

(b) *$M(1_G; \Sigma)$ is the identity matrix, $M(\phi; \Sigma) = PM(\phi; \Delta)P^{-1}$, and $M(\phi; \Sigma, \Delta) = M(\phi; \Sigma)P = PM(\phi; \Delta)$ for all $\phi \in$* $\operatorname{cf}(G)$.

(c) $\{M(\sigma;\Sigma)|\sigma\in\Sigma\}$ *is a basis of* $M(G;\Sigma)$; *the eigenvalues of the arbitrary* (G,Σ)*-circulant* $\sum_{\sigma\in\Sigma}a_\sigma M(\sigma;\Sigma)$ *are* $\sum_{\sigma\in\Sigma}a_\sigma\sigma(C_i)$, $i=1,2,\dots,k$.

(d) *If* Σ *is orthonormal, then*:

(1) $M(\bar{\phi};\Sigma)=(M(\phi;\Sigma))^*$ *for all* $\phi\in \mathrm{cf}(G)$, *and* $M(G;\Sigma)$ *is closed under the* $*$*-operation and under M-P inversion. In particular each* $(G;\Sigma)$*-circulant is normal.*

(2) *If* $[\sigma_i\sigma_j,\sigma_k]$ *is real for all* $\sigma_i,\sigma_j,\sigma_k\in\Sigma$, *then* $M(G,\Sigma)$ *is closed under transposition.*

(3) *Let* $X=X(G;\Sigma)$, $D=(1/|G|)\mathrm{diag}(|C_1|,|C_2|,\dots,|C_k|)$, *and* $V=XD(XD)^*$. *Then* $XDX^*=I$, *and the map* $\phi\to M(\phi;\Sigma)$ *is an isometry relative to the usual inner product in* $\mathrm{cf}(G)$ *and the inner product* $[A,B]=\mathrm{tr}(AVB^*)$ *in* $M(G;\Sigma)$.

Proof. Proposition 1.1 implies all parts except (d). So assume that Σ is orthonormal. Let Γ be some fixed ordering of $\mathrm{Irr}(G)$, and Q^t the transition matrix from Γ to Σ. As both Γ and Σ are orthonormal, Q is unitary. Let $Y=Y(G;\Gamma)$, the character table of G. By [1, Theorem 1.4(c)] $YDY^*=I$, and by part (a) $X=QY$. Hence, $XDX^*=QYDY^*Q^*=QQ^*=I$. The proof of the rest of (d) is exactly like the proof of the corresponding parts of [1] (i.e., Proposition 1.2(b); Corollary 1.3(a), (c), and Theorem 1.4(f) of [1]).

PROPOSITION 1.3. *Let* A, k, $\mathscr{E}$, Σ, Δ, P^t, *and* $X=X(A;\Sigma)$ *be as in Proposition* 1.1 (*respectively: let* G, k, Σ, Δ, P^t, *and* $X=X(G;\Sigma)$ *be as in Corollary* 1.2). *Let* M *be a complex* $k\times k$ *matrix. Then*:

I. *The following are equivalent*:

(a) $M\in M(A;\Sigma)$ (*respectively*: M *is a* $(G;\Sigma)$*-circulant*);
(b) M *commutes with* $M(\sigma;\Sigma)$ *for all* $\sigma\in\Sigma$;
(c) $M=XDX^{-1}$ *for some diagonal complex matrix* D;
(d) $M=PCP^{-1}$ *for some* $C\in M(A;\Delta)$ (*respectively: for some* $(G;\Delta)$*-circulant* C);
(e) M *is a polynomial in each* $M(a;\Sigma)$ *that has* k *distinct eigenvalues* (*respectively*: M *is a polynomial in* $M(\phi;\Sigma)$ *for every* $\phi\in\mathrm{cf}(G)$ *that has* k *distinct values on* G).

II. *The following are equivalent*:

(a) $M\in M(A;\Sigma,\Delta)$ (*respectively*: M *is a* $(G;\Sigma,\Delta)$*-circulant*);
(b) $M=PN$ *for some* $N\in M(A;\Delta)$ (*respectively: for some* $(G;\Delta)$*-circulant* N);
(c) $M=PYDY^{-1}$ *for some diagonal complex matrix* D *and* $Y=X(A;\Delta)$ (*respectively*: $Y=X(G;\Delta)$);
(d) $PM(\delta;\Delta)P^{-1}\cdot M=M\cdot M(\delta;\Delta)$ *for all* $\delta\in\Delta$.

Proof. Only the statements on A need proof.

I: (a) $\Leftrightarrow$ (d) follows from Proposition 1.1(c). Both $M(A;\Sigma)$ and $M' = \{XDX^{-1} | D$ is a diagonal $k \times k$ complex matrix$\}$ are subspaces of dimension k. By Proposition 1.1(a) we have that $M(A;\Sigma) \subseteq M'$, so $M(A;\Sigma) = M'$ and (a) $\Leftrightarrow$ (c) follows. Since $M(A;\Sigma)$ is a commutative algebra, we have that (e) $\Rightarrow$ (a) $\Rightarrow$ (b). We need to show that (b) $\Rightarrow$ (e). Assuming (b), we know that M commutes with all members of $M(A;\Sigma)$; in particular it commutes with each $M(a;\Sigma)$ with k distinct eigenvalues. Such $M(a;\Sigma)$ is nonderogatory, so that Theorem 6.A.2 on p. 232 of [2] implies (e).

II: (a) $\Leftrightarrow$ (b) is a consequence of Proposition 1.1(b), (c), and (b) $\Leftrightarrow$ (c) follows from I. Now (b) is equivalent to $P^{-1}M \in M(A;\Delta)$, which is equivalent (by I) to $P^{-1}M \cdot M(\delta;\Delta) = M(\delta;\Delta) \cdot P^{-1}M$ for all $\delta \in \Delta$, which is (d). ∎

Remark. Keeping the above notation and assuming that Σ is orthonormal, the *M-P* inverse of the $(G;\Sigma)$-circulant $X \cdot \mathrm{diag}(a_1, a_2, \ldots, a_k) \cdot X^{-1}$ is $X \cdot \mathrm{diag}(a_1^{\div}, a_2^{\div}, \ldots, a_k^{\div}) \cdot X^{-1}$, where

$$\alpha^{\div} = \begin{cases} 0 & \text{if } \alpha = 0, \\ 1/\alpha & \text{if } \alpha \neq 0. \end{cases}$$

This follows as in Corollary 1.6 of [1]. The other parts of Corollary 1.6 of [1], as well as Corollary 1.7 of [1], can be generalized straightforwardly to $(G;\Sigma)$-circulants.

Remark (On noncommutative semisimple algebras). Structure theory (the Wedderburn–Artin–Burnside theory) for finite-dimensional semisimple (not necessarily commutative) algebras A over an algebraically closed field K is known. In fact, such an A is isomorphic to a direct sum of r complete matrix algebras over K, where r is the dimension of the center of A over K. Also, r is the number of inequivalent irreducible representation of A over K. See [5, Chapter V]. In this context, the fact that representing matrices of finite period can be diagonalized is cited there as an application of Mascke's theorem (see p. 20 of [3]).

II. SPECIAL CASES—EXAMPLES

If G is a finite group with a known character table M, then much information on $(G;\Sigma)$-circulants (e.g., eigenvalues, eigenvectors, etc.) can be read from M and some transition matrix. For this reason, in all but one of the following examples of $M(A;\Sigma)$ the algebra A will be $cf(G)$.

(a) *Circulants, Skew-Circulants, $\{k\}$-Circulants, and Others*

Let $G = C_n = \langle u \rangle$ be a cyclic group of order n with a generator u. Let $\chi_j \in \mathrm{Irr}(G)$ be defined by $\chi_j(u) = \omega^j$, $0 \leqslant j \leqslant n-1$, where $\omega = e^{2\pi i/n}$. Let Δ be the following ordering of $\mathrm{Irr}(G)$: $\Delta = \{\chi_0, \chi_1, \chi_2, \dots, \chi_{n-1}\}$. Clearly, $\chi_0 = 1_G$, $\chi_1^n = \chi_0$, $\chi_i = \chi_1^i$ for $i = 0, 1, \dots, n-1$. Next, fix a nonzero complex number c, set $c^n = k$, and let $\sigma_i = c^i \chi_i$ for $i = 0, 1, \dots, n-1$. Then $\Sigma = \{\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$ is a basis of $\mathrm{cf}(G)$, and

$$\sigma_i \sigma_j = \begin{cases} \sigma_{i+j} & \text{if} \quad i + j < n, \\ k\sigma_{i+j-n} & \text{if} \quad i + j \geqslant n. \end{cases}$$

Let $\phi \in \mathrm{cf}(G)$ be written relative to Σ: $\phi = \sum_{j=0}^{n-1} c_j \sigma_j$, where $c_j \in \mathbb{C}$. Then $\phi\sigma_i = \sum_{j=0}^{n-1} c_j(\sigma_j\sigma_i)$, and so the $(i, (i+j) \pmod n)$th entry of $M(\phi; \Sigma)$ is c_j if $i + j < n$ and is kc_j if $i + j \geqslant n$. Consequently,

$$M(\phi;\Sigma) = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & \cdots & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & c_2 & \cdots & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} \\ kc_{n-3} & kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ kc_1 & kc_2 & kc_3 & kc_4 & \cdots & c_0 \end{bmatrix}$$

$$= \mathrm{circ}_k(c_0, c_1, c_2, \dots, c_{n-1}) \tag{$*$}$$

It follows that $M(G; \Sigma)$ is the family of all matrices of the form $(*)$. In particular: $M(G; \Sigma)$ is the set of all circulants if $k = c = 1$; $M(G; \Sigma)$ is the set of all skew-circulants if $k = -1$ and $c = e^{\pi i/n}$; $M(G; \Sigma)$ is the set of all $\{k\}$-circulants if $k = e^{i\alpha}$ and $c = e^{i\alpha/n}$ (see [2, pp. 66, 83–84] for definitions). We note that in the above mentioned three special cases (as well as whenever $|c| = 1$) the basis Σ is orthonormal.

Note that

$$M(\sigma_0; \Sigma) = I_n \quad \text{and} \quad M(\sigma_1; \Sigma) = \eta_k = \begin{pmatrix} 0 & I_{n-1} \\ kI_1 & 0 \end{pmatrix},$$

where I_s is an identity matrix of order s. Also $M(\sigma_i; \Sigma) = (M(\sigma_1; \Sigma))^i = (\eta_k)^i$ for $i = 0, 1, \dots, n-1$. Hence, an $n \times n$ matrix commutes with all matrices in

$\{M(\sigma;\Sigma)|\sigma\in\Sigma\}$ if and only if it commutes with η_k. Moreover, as $\phi = \sum_{i=0}^{n-1}c_i\sigma_i = \sum_{i=0}^{n-1}c_i\sigma_1^i$, we get that $\mathrm{circ}_k(c_0,c_1,c_2,\ldots,c_{n-1}) = \sum_{i=0}^{n-1}c_i(\eta_k)^i$.

Next, let $C_i = \{u^i\}$, $i = 0,1,2,\ldots,n-1$, be the conjugacy classes of G. Then $X(G;\Delta) = (\omega^{ij})_{i,j=0,1,\ldots,n-1} = \sqrt{n}\,F_n^* = \sqrt{n}\,F^*$ (see [2, p. 32] for the definition of F^*). The transition matrix from Δ to Σ is $P = P^t = \mathrm{diag}(1,c,c^2,c^3,\ldots,c^{n-1})$, and by Corollary 1.2(a) $X = X(G;\Sigma) = \sqrt{n}\,PF^* = \sqrt{n}\,(c^i\omega^{ij})_{i,j=0,1,\ldots,n-1}$. Note that if $|c| = 1$, then $P^* = P^{-1}$. Finally, by Corollary 1.2(c) the eigenvalues of $\mathrm{circ}_k(c_0,c_1,c_2,\ldots,c_{n-1}) = M(\phi;\Sigma)$ (where $\phi = \sum_{i=0}^{n-1}c_i\sigma_i$) are $\sum_{i=0}^{n-1}c_i\sigma_i(u^j) = \sum_{i=0}^{n-1}c_i(c\omega^j)^i$, $j = 0,1,2,\ldots,n-1$. These observations show that the following results from [2] are special cases of the results of our Section I: All results of pp. 66, 67, 68, 72, 73; Theorem 3.2.4; Problems 9, 10 of p. 70; Problems 18, 19 of p. 81; all results of pp. 83, 84, 85, including, of course, all the problems; Theorem 3.3.1 and its corollary; Results 3.4.10 and 3.4.21; the corollaries of pp. 102 and 103; and others.

Replacing the above Σ with other bases of $\mathrm{cf}(G)$ (preferably orthonormals) yield many other circulant-like families of matrices. For example, one can multiply each of the χ_i's by a root of unity at will, to get an orthonormal Σ and a new family.

(b) *Retrocirculants, some g-Circulants, and Others*

Let $G = C_n$, Δ,χ_i be as in Section II(a). For every base $\Gamma = \{\gamma_0,\gamma_1,\ldots,\gamma_{n-1}\}$ of $\mathrm{cf}(G)$ and every permutation τ on $0,1,2,\ldots,n-1$, let $\tau(\Gamma) = \{\gamma_0',\gamma_1',\ldots,\gamma_{n-1}'\}$ where $\gamma_i' = \gamma_{\tau(i)}$ for all $i = 0,1,2,\ldots,n-1$. Then $\tau(\Gamma)$ is a basis of $\mathrm{cf}(G)$. The transition matrix from Γ to $\tau(\Gamma)$ is $(Q_\tau)^t$, where Q_τ is the permutation matrix associated with τ. By Corollary 1.2(b), $M(G;\tau(\Gamma),\Gamma) = \{Q_\tau\cdot M(\phi;\Gamma)|\phi\in\mathrm{cf}(G)\} = \{M(\phi;\tau(\Gamma))\cdot Q_\tau|\phi\in\mathrm{cf}(G)\}$.

Now, if

$$\tau = \tau_g = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & g & 2g & 3g & \cdots & (n-1)g \end{pmatrix},$$

where products are modulo n and g an integer with $(n,g) = 1$, then $M\big(G;\tau_g(\Delta),\Delta\big)$ is the set of all g-circulants and $Q_\tau = Q_g = \text{g-circ}(1,0,0,\ldots,0)$ (see [2, pp. 155, 161] for definitions). Also, if

$$\tau = \tau_{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & n-1 & n-2 & n-3 & \cdots & 1 \end{pmatrix},$$

then $M(G;\tau_{-1}(\Delta),\Delta)$ is the set of all retrocirculants and $Q_\tau = Q_{-1} = \Gamma$ of p. 28 of [2] (see also p. 156 of [2]). Note that in these two special cases,

$M(G;\Delta)$ is the set of all $n\times n$ circulants and so $M(G;\tau(\Delta),\Delta)=\{Q^t\cdot C|C$ is a circulant$\}$. Let π be as on p. 27 of [2]; then $\pi=M(\chi_1;\Delta)$ [or η_1 for $k=\sigma=1$ in Section II(a)], and $\pi^g=M(\chi_1^g;\Delta)$ commutes with a matrix M if and only if each matrix of $\{M(\chi;\Delta)|\chi\in\Delta\}$ commutes with M. Also, it can be seen (by considering the corresponding permutations) that $Q_g\cdot\pi^g\cdot(Q_g)^{-1}=\pi$ for $g=-1$ or g with $(g,n)=1$. Now, Theorem 5.1.1, Theorem 5.1.7, and its corollaries of [2] follow from our Proposition 1.3 [when $g=-1$ or $(g,n)=1$] as special cases. Some other results of Chapter 5.1 of [2] are special cases of general results on $(G;\tau(\Gamma),\Gamma)$-circulants with an orthonormal Γ; for example, Theorem 5.1.5 of [2] and its corollary are valid for such $(G;\tau(\Gamma),\Gamma)$-circulants.

(c) *A Nonabelian example and a different algebra*

Let $G=S_3$, the symmetric group on 3 letters. Let $\Lambda=\{\chi_1,\chi_2,\chi_3\}$ be as in Example 3 of [2, Section III]. Then (by Example 3 of [2]) $M(G;\Lambda)$ is the set of all matrices of the form

$$\begin{bmatrix} a & b & c\\ b & a & c\\ c & c & a+b+c\end{bmatrix},$$

and

$$X(G;\Lambda)=\begin{bmatrix} 1 & 1 & 1\\ 1 & -1 & 1\\ 2 & 0 & -1\end{bmatrix}.$$

Set $\Gamma=\{\gamma_1,\gamma_2,\gamma_3\}$, where $\gamma_1=\chi_1$, $\gamma_2=-\chi_2$, $\gamma_3=i\chi_3$. The transition matrix from Λ to Γ is $P=P^t=\operatorname{diag}(1,-1,i)$, and so

$$M(G;\Gamma)=\left\{\begin{bmatrix} a & -b & -ic\\ -b & a & ic\\ ic & -ic & a+b+c\end{bmatrix}\middle| a,b,c\in\mathbb{C}\right\}$$

$$=\left\{\begin{bmatrix} a & b & c\\ b & a & -c\\ -c & c & a-b-ic\end{bmatrix}\middle| a,b,c\in C\right\},$$

$$M(G;\Gamma,\Lambda)=\left\{\begin{bmatrix} a & b & c\\ -b & -a & -c\\ ic & ic & i(a+b+c)\end{bmatrix}\middle| a,b,c\in\mathbb{C}\right\},$$

and

$$X(G;\Gamma)=\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 2i & 0 & -i \end{bmatrix}.$$

Another FDSSC algebra, A, associated with a finite group G is the center of the group algebra G. A natural basis of A, Σ, consists of the class sums of G. For $G = S_3$, for example, $M(A;\Sigma)$ (with a proper ordering of the class sums) is the set of all matrices of the form

$$\begin{bmatrix} a & b & c \\ 3b & a+2c & 3b \\ 2c & 2b & a+c \end{bmatrix},$$

and

$$X(A;\Sigma)=\begin{bmatrix} 1 & 1 & 1 \\ 3 & -3 & 0 \\ 2 & 2 & -1 \end{bmatrix}$$

diagonalizes each matrix of $M(A;\Sigma)$.

Families of matrices as in this example can be constructed from every finite group with a known character table.

(d) *Direct products*

The following fact from the next section can be used to construct new circulant-like families from old ones: Let H and K be two finite groups, and Σ_1, Σ_2 bases for $\mathrm{cf}(H)$ and $\mathrm{cf}(K)$ respectively. Then, there exists a basis Σ of $\mathrm{cf}(H \times K)$ for which $M(H \times K;\Sigma)$ is the set of all block $(H;\Sigma_1)$-circulants in which the blocks are $(K;\Sigma_2)$-circulants. Also, $X(H \times K;\Sigma) = X(H;\Sigma_1) \otimes X(K;\Sigma_2)$, where $\otimes$ is the Kronecker product. See the next section for the precise statement.

In Example 2 of [1, Section III] this fact is used to show how block circulants with circulant blocks of arbitrary level arise from abelian groups. We give another example here.

Let $G_1 = C_4$ and Σ be as in Section II(a) with respect to $k, c \in \mathbf{C}$ with $c^4 = k$ and $|k| = 1$. Let $G_2 = G_3 = S_3$ and Γ, Λ be as in Section II(c). Finally, let $G_4 = C_n$ with Δ as in Section II(a). Let $G = G_1 \times G_2 \times G_3 \times G_4$. Then for some Σ', $M(G;\Sigma')$ is the algebra of all block matrices of the form

$$\begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ kA_3 & A_0 & A_1 & A_2 \\ kA_2 & kA_3 & A_0 & A_1 \\ kA_1 & kA_2 & kA_3 & A_0 \end{bmatrix},$$

where each A_i, $i=0,1,2,3$, is a block matrix of the form

$$\begin{bmatrix} A & B & C \\ B & A & -C \\ -C & C & A-B-iC \end{bmatrix},$$

where A, B, C are block matrices of the form

$$\begin{bmatrix} E & F & G \\ F & E & G \\ G & G & E+F+G \end{bmatrix},$$

where E, F, and G are arbitrary $n\times n$ circulants. Also,

$$X(G;\Sigma) = 2\operatorname{diag}(1,c,c^2,c^3)\,F_4^*\otimes\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 2i & 0 & -i \end{bmatrix}$$

$$\otimes\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}\otimes\sqrt{n}\,F_n^*$$

diagonalizes each element of $M(G;\Sigma)$.

We remark that $(G;\Sigma,\Delta)$-circulants can be "direct-multiplied" as well. See the next section.

III. BLOCK $(G;\Sigma,\Delta)$-CIRCULANTS

NOTATION.

(1) The Kronecker product of the $\mathbb{C}$-vector spaces of matrices M_1 and M_2 is the space spanned by $\{C\otimes D \mid C\in M_1,\ D\in M_2\}$; it will be denoted by $M_1\otimes M_2$. Clearly $M_1\otimes M_2$ is a $\mathbb{C}$-vector space, and it is a $\mathbb{C}$-algebra if both M_1 and M_2 are C-algebras.

(2) Let $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ and $\Delta = \{\delta_1, \delta_2, \dots, \delta_k\}$ be two bases of the FDSSC algebra A. Recall that the (Σ, Δ)-constants of A are the k^3 numbers a_{ijr} defined by $\delta_j \sigma_i = \sigma_i \delta_j = \sum_{r=1}^{k} a_{ijr} \delta_r$. Assume that $1 = \sigma_{m_0}$ for some m_0. We call m_0 the leading index of Σ. The leading row of a $k \times k$ matrix is the m_0th row. let $B = (B_{ij})$ be a block $k \times k$ matrix with equal-size square blocks. We say that B is a block $(A; \Sigma, \Delta)$-matrix if $B_{ij} = \sum_{t=1}^{k} B_{m_0 t} \cdot a_{itj}$, for all i, j. This, in fact, says that the B_{ij}'s satisfy the same "relations" as the entries of the $M(A; \Sigma, \Delta)$ matrix $M(b; \Sigma, \Delta) = (b_{ij})$, because $b = b \cdot \sigma_{m_0} = \sum_{t=1}^{k} b_{m_0 t} \delta_t$ implies that $b \cdot \sigma_i = \sum_{t=1}^{k} b_{m_0 t}(\delta_t \cdot \sigma_i) = \sum_{j=1}^{k} (\sum_{t=1}^{k} b_{m_0 t} a_{itj}) \delta_j$, so that $b_{ij} = \sum_{t=1}^{k} b_{m_0 t} a_{itj}$ (see Section II of [1] for more details). We note that if $\Sigma = \Delta$ then $a_{itj} = a_{tij}$ for all t, i, j. If $A = \mathrm{cf}(G)$ for a finite group G, we replace A by G in the above notation, and a block $(G; \Sigma, \Delta)$-matrix will be called a block $(G; \Sigma, \Delta)$-circulant.

(3) Let $G = H \times K$ be a direct product of the finite groups H and K. For each $\eta \in \mathrm{cf}(H)$ and $\phi \in \mathrm{cf}(K)$ define $\eta \times \phi \in \mathrm{cf}(G)$ by $(\eta \times \phi)(x, y) = \eta(x)\phi(y)$ for all $x \in H$, $y \in K$. It is well known that $\mathrm{Irr}(G) = \{\eta \times \phi | \eta \in \mathrm{Irr}(H), \phi \in \mathrm{Irr}(K)\}$. Finally, if $\Phi = \{\phi_1, \phi_2, \dots, \phi_r\}$ is a basis of $\mathrm{cf}(H)$ and $\Psi = \{\psi_1, \psi_2, \dots, \psi_s\}$ a basis of $\mathrm{cf}(K)$, we denote by $\Phi \times \Psi$ the set $\{\phi_i \times \psi_j | 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s\}$ where the $\phi_i \times \psi_j$'s are ordered lexicographically. We will see that $\Phi \times \Psi$ is a basis of $\mathrm{cf}(G)$.

PROPOSITION 3.1. *Let $G = H \times K$ be a direct product of the finite groups H and K. Let Σ_1 and Δ_1 be bases of* $\mathrm{cf}(H)$, *and Σ_2 and Δ_2 bases of* $\mathrm{cf}(G)$. *Set $\Sigma = \Sigma_1 \times \Sigma_2$ and $\Delta = \Delta_1 \times \Delta_2$. Then*:

(a) $M(\eta \times \theta; \Sigma, \Delta) = M(\eta; \Sigma_1, \Delta_1) \otimes M(\theta; \Sigma_2, \Delta_2)$ *for all* $\eta \in \mathrm{cf}(H)$ *and* $\theta \in \mathrm{cf}(K)$.

(b) $M(G; \Sigma, \Delta) = M(H; \Sigma_1, \Delta_1) \otimes M(K; \Sigma_2, \Delta_2)$.

(c) $X(G; \Sigma) = X(G; \Sigma_1) \otimes X(G; \Sigma_2)$.

(d) *Assume that $1_G \in \Sigma_1$. Then a matrix is a $(G; \Sigma, \Delta)$-circulant if and only if it is a block $(H; \Sigma_1, \Delta_1)$-circulant in which the blocks are all $(K; \Sigma_2, \Delta_2)$-circulants.*

Proof. Let β and γ be orderings of $\mathrm{Irr}(H)$ and $\mathrm{Irr}(K)$ respectively, as follows: $\beta = \{\eta_1, \eta_2, \dots, \eta_r\}$, $\gamma = \{\theta_1, \theta_2, \dots, \theta_s\}$. Set $\Sigma_1 = \{\sigma_1, \sigma_2, \dots, \sigma_r\}$ and $\Sigma_2 = \{\delta_1, \delta_2, \dots, \delta_s\}$, and let P^t and Q^t be the transition matrices from β to Σ_1 and from γ to Σ_2, respectively. Write $P = (p_{ij})$ and $Q = (q_{ij})$. Then $\sigma_u \times \delta_v = (\sum_{i=1}^{r} p_{ui} \eta_i) \times (\sum_{j=1}^{s} q_{vj} \theta_j) = \sum_{i=1}^{r} \sum_{j=1}^{s} p_{ui} q_{vj} (\eta_i \times \theta_j)$. This implies that $(P \otimes Q)^t$ is the transition matrix from $\alpha = \beta \times \gamma$ to $\Sigma = \Sigma_1 \times \Sigma_2$. As $P \otimes Q$ is nonsingular, Σ is a basis of $\mathrm{cf}(G)$. Applying Corollary 1.2(a) and (b) to Theorem 2.1(a),(c) of [1] (note that $\alpha = \beta \times \gamma$ in Theorem 2.1 of [1]), we get $M(\eta \times \theta; \Sigma) = (P \otimes Q) M(\eta \times \theta; \alpha)(P \otimes Q)^{-1} = (P \otimes Q)[M(\eta; \beta) \otimes$

$M(\theta;\gamma)](P\otimes Q)^{-1} = [PM(\eta;\beta)P^{-1}]\otimes[QM(\theta;\gamma)Q^{-1}] = M(\eta;\Sigma_1)\otimes M(\theta;\Sigma_2)$. So, for all $\eta\in \mathrm{cf}(H)$ and $\theta\in\mathrm{cf}(K)$,

$$M(\eta\times\theta;\Sigma) = M(\eta;\Sigma_1)\otimes M(\theta;\Sigma_2). \qquad (*)$$

Also, $X(G;\Sigma) = (P\otimes Q)X(G;\alpha) = (P\otimes Q)[X(H;\beta)\otimes X(K;\gamma)] = [P\cdot X(H;\beta)]\otimes[Q\cdot X(K;\gamma)] = X(H;\Sigma_1)\otimes X(K;\Sigma_2)$, which is (c). (See [2, pp. 22–23] for the properties of $\otimes$ that were used).

Let R^t and T^t be the transition matrices from Δ_1 to Σ_1 and from Δ_2 to Σ_2 respectively. As above, $(R\otimes T)^t$ is the transition matrix from $\Delta=\Delta_1\times\Delta_2$ to $\Sigma=\Sigma_1\times\Sigma_2$. Applying Corollary 1.2(b) to $(*)$, we obtain $M(\eta\times\theta;\Sigma,\Delta) = M(\eta\times\theta;\Sigma)(R\otimes T) = [M(\eta;\Sigma_1)\otimes M(\theta;\Sigma_2)](R\otimes T) = [M(\eta;\Sigma_1)R]\otimes[M(\theta;\Sigma_2)T] = M(\eta;\Sigma_1,\Delta_1)\otimes M(\theta;\Sigma_2,\Delta_2)$, which is (a). Next, the mapping $\rho\to M(\rho;\Sigma,\Delta)$ is linear, and so $M(G;\Sigma,\Delta)$ is a $\mathbb{C}$-vector space spanned by matrices of the form $M(\eta\times\theta;\Sigma,\Delta)$, $\eta\in\mathrm{cf}(H)$, $\theta\in\mathrm{cf}(G)$. This implies part (b). Finally, part (d) follows from part (b) and the next more general statement. ∎

Proposition 3.2. *Let A_1 and A_2 be two FDSSC algebras, and Σ_i,Δ_i two bases of A_i, $i=1,2$. Assume that $1_\epsilon\Sigma_1$. Then a matrix belongs to $M(A_1;\Sigma_1,\Delta_1)\otimes M(A_2;\Sigma_2,\Delta_2)$ if and only if it is a block $(A_1;\Sigma_1,\Delta_1)$-matrix in which all the blocks belong to $M(A_2;\Sigma_2,\Delta_2)$.*

Proof. Set $\Sigma_1=\{\eta_1,\eta_2,\dots,\eta_r\}$, $\Delta_1=\{\delta_1,\delta_2,\dots,\delta_r\}$, and $s=\dim(A_2)$. Now read the proof of Theorem 2.3 of [1] with the following changes: (1) Replace $\mathrm{cf}(H)$ by A_1; $\mathrm{cf}(K)$ by A_2; β by Σ_1; γ by Σ_2; $M^\beta(\tau_u)$ by $M(\tau_u;\Sigma_1,\Delta_1)$; $M^\gamma(\phi_v)$ by $M(\phi_v;\ \Sigma_2,\Delta_2)$; $[\eta_t\eta_i,\eta_j]=[\eta_i\eta_t,\eta_j]$ by the (Σ_1,Δ_1)-constant a_{itj}; $M^\beta(\eta_t)$ by $M(\delta_t;\Sigma_1,\Delta_1)$. (2) For a matrix D replace "D is a (H,β)-circulant [or $D\in M(H;\beta)$]," "D is a (K,γ)-circulant [or $D\in M(K;\gamma)$]," "D is a (G,α)-circulant [or $D\in M(G,\alpha)$]," and "block (H,β)-circulant" by "$D\in M(A_1;\Sigma_1,\Delta_1)$," "$D\in M(A_2;\Sigma_2,\Delta_2)$," "$D\in M(A_1;\Sigma_1,\Delta_1)\otimes M(A_2;\Sigma_2,\Delta_2)$," and "block (A_1,Σ_1,Δ_1)-matrix," respectively. ∎

Remarks.

(1) Proposition 3.1 can be stated for a direct product of more than two groups, like Corollary 2.4 of [1].

(2) One can define block $(A;\Sigma,\Delta)$-matrices without assuming that $1\in\Sigma_1$, but this assumption simplifies the definition.

I would like to thank the referee for pointing out that by naturally extending the theory of this paper to the regular representation of the group algebra, results of Wang [6] can be obtained as a consequence.

REFERENCES

1 D. Chillag, Generalized circulants and class functions of finite groups, *Linear Algebra Appl.* 93:191–208 (1987).
2 P. J. Davis, *Circulant Matrices*, Wiley, New York, 1979.
3 I. M. Isaacs, *Character Theory of Finite Groups*, Academic, 1976.
4 D. A. Suprunenko and R. I. Tyshkevich, *Commutative Matrices*, Academic, New York, 1968.
5 B. Huppert, *Endliche Gruppen I*, Springer, New York, 1967.
6 K. Wang, On the generalization of circulants, *Linear Algebra Appl.* 25:197–218 (1979); On the group matrices for the generalized dihedral group, *ibid.* 39:83–89 (1981).

Received 9 March 1987; final manuscript accepted 4 December 1987